# Robust Design in Multi-Objective Optimization

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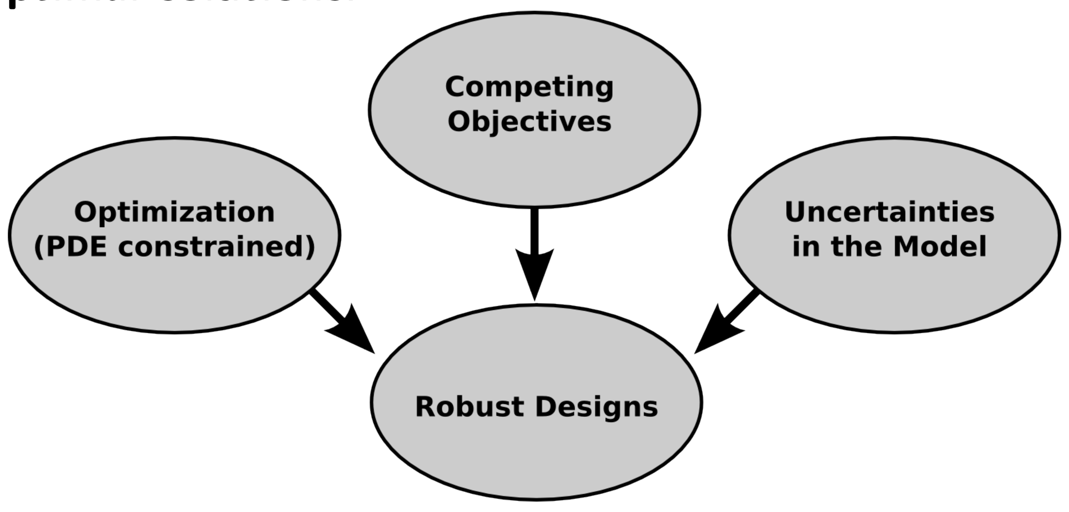
## Motivation

**Engineering design** often involves the optimization of different **competing objectives**. The aim is to find a set of solutions that fulfil the concept of Pareto optimality. A further significant step to realistic multi-objective designs is to take into account **uncertainties** for finding **robust optimal solutions**.




## Multi-Objective Optimization (MOO)

The aim of multi-objective optimization algorithms is to find a representative subset of **Pareto optimal** solutions. A solution is Pareto optimal if it is only possible to improve one objective function at the expense of worsening at least one other. So far, multi-objective robust design is mainly treated in an evolutionary context (see e.g. [1]). Multi-objective evolutionary algorithms are usually computationally expensive and slow in terms of convergence. We make use of the deterministic Epsilon-Constraint Method [2].

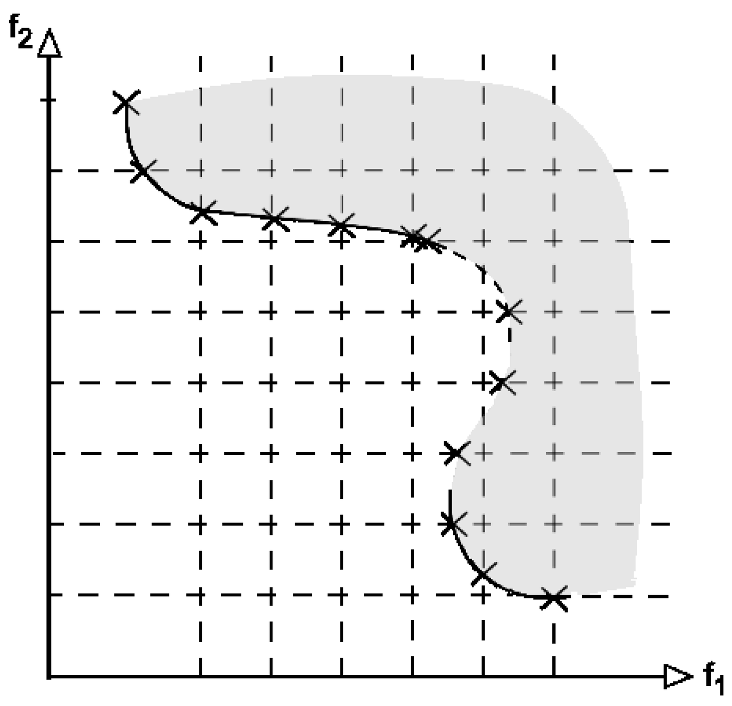
### Epsilon-Constraint Method

The concept of the Epsilon-Constraint Method is to optimize one objective function while imposing inequality constraints on the remaining competing objective functions. The constraints as well as the objective function to be optimized are varied to find different Pareto optimal solutions that are evenly distributed.

The resulting problem for $k$ objective functions with the PDE constraint $c = 0$ reads

$$\begin{aligned} \min_{y,u} \quad & f_s(y,u) \\ \text{s.t.} \quad & c(y,u) = 0 \\ & f_i(x) \leq f_i^{(j)} \\ & \forall\, i \in \{1, \dots, k\} \quad i \neq s \end{aligned} \qquad (1)$$

The figure to the right depicts the procedure of the Equality-Constraint Method $(f_i(x) = f_i^{(j)})$ for two objectives.



Scanning procedure with Equality-Constraint Method

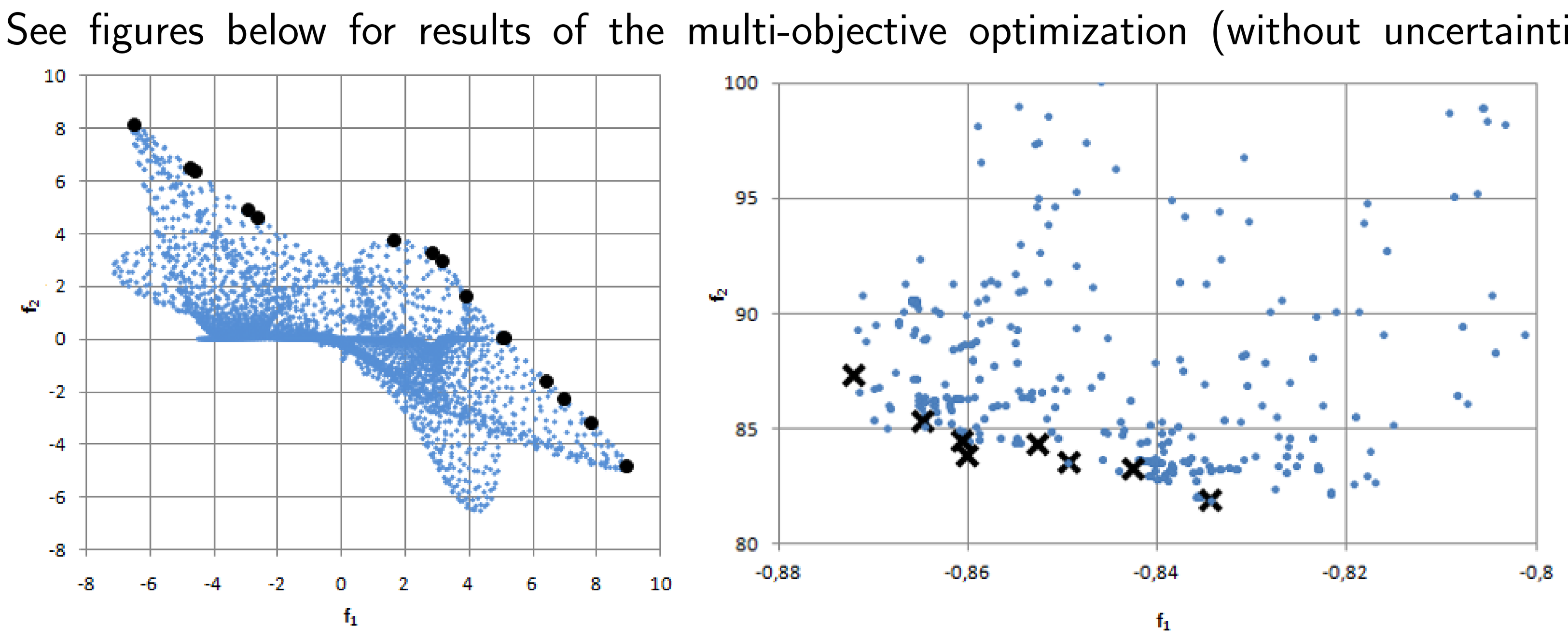
The black line is denoted as the Pareto optimal front. There also exist points on the containing surface that are only locally Pareto optimal (dashed line) or not Pareto optimal (no line). An advantage of the Epsilon-Constraint Method is that all unique solutions to (1) are globally Pareto optimal for any upper bound.

To enhance the chance of finding a global optimum the resulting **single-objective optimization** problems are solved with a **hybrid approach** combining

- a genetic algorithm (GA) on a response surface with
- a gradient-based quasi-Newton method (Ipopt) using the approximated optimum found by the GA as a starting point.

## MOO Results

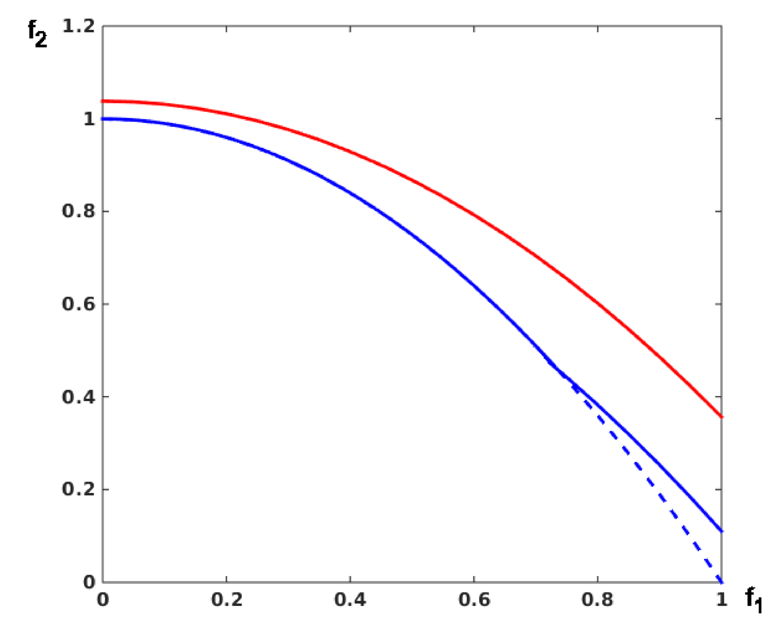
See figures below for results of the multi-objective optimization (without uncertainties).


Results for a complicated test case (left) and shape optimization of exhaust after-treatment system (right)

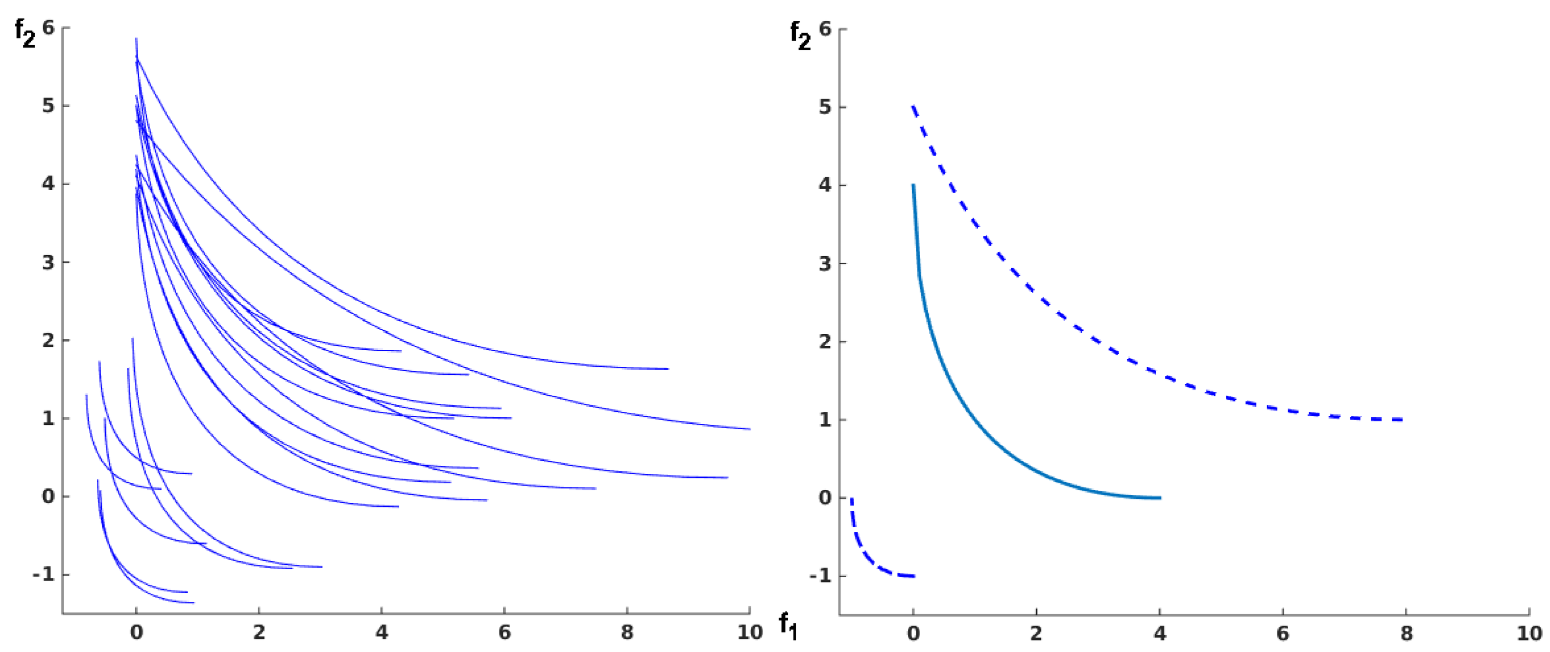
## Robustness in MO context

We consider a solution robust if it is not very sensitive to aleatory uncertainties $\omega$. In multi-objective design **different concepts** for finding robust solutions can be considered:

- expectation-based approach:
$$\min_{y,u} \ \text{Exp}(\mathbf{F}(y,u,x(\omega))) \qquad (2)$$
or
$$\min_{y,u} \mathbf{F}(y,u,\bar{x}) \qquad (3)$$
$$\text{s.t.} \|\text{Exp}(\mathbf{F}) - \mathbf{F}\| \leq \nu,$$
- expectation-variance-based approach:
$$\min_{y,u} \ (\text{Exp}(\mathbf{F}), \text{Var}(\mathbf{F})) \qquad (4)$$
or
$$\min_{y,u} \mathbf{F}(y,u,\bar{x}) \qquad (5)$$
$$\text{s.t.} \|\text{Var}(\mathbf{F})\| \leq \mu,$$



Example of an expected Pareto front (red, according to (2)) and a robust Pareto front (blue, (3)) with design uncertainties



Sampling of Pareto fronts (left) and resulting expected Pareto front (–) with trust region borders (- -) (right) with parameter uncertainties

- trust or confidence region for the Pareto optimal front (see figure above),
- or other concepts (probability of dominance, local sensitivity region).

## Uncertainty Quantification

There exist different methods to propagate uncertainties $\omega$ in the model. As the costs of a multi-objective optimization are very high, it is important to use efficient approaches. We make use of a **non-intrusive polynomial chaos** approach. In this approach the stochastic objective function is expanded in terms of polynomials $\Phi_i$ that are orthogonal with respect to the density function of the input random variables $x(\omega)$:

$$f(y,u,x(\omega)) \approx \sum_{i=1}^{M} f_i(y,u)\Phi_i(x), \quad f_i(y,u) = \gamma_i \mathbb{E}(f(y,u,x(\omega))\Phi_i).$$

The non-intrusive approach results in a multiple set-point problem. The computational effort can be reduced by using sparse grids for the quadrature points. In Schillings et al. [3] the method is used for single-objective aerodynamic robust design with one-shot optimization.

## Future Work

- Application to aerodynamic shape optimization with SU2. Promising objectives: drag/lift, moment of inertia, aero-elasticity?
- One-shot approach using algorithmic differentiation (already prepared) for the single-objective optimization. How to treat the inequality constraints?
- Comparison of concepts of robustness. How to average (approximated) level sets?

### References

[1] K. Deb, H. Gupta, Searching for Robust Pareto-Optimal Solutions in Multi-Objective Optimization. Lecture Notes in Computer Science, 3410, 150-164, 2005.

[2] S. A. Marglin, Public investment criteria: Benefit-cost analysis for planned economic growth. M.I.T Press, Cambridge, Massachusetts, 1967.

[3] C. Schillings, S. Schmidt, V. Schulz, Efficient shape optimization for certain and uncertain aerodynamic design. Computers and Fluids, 76, 78-87, 2011.